Indian J. Phys. **68B** (6), 539 – 543 (1994)

I J P B
— an international journal

On the spherically symmetric static solutions of Brans-Dicke field equations in vacuum

Kamal K Nandi and Anwarul Islam*

Department of Mathematics, University of North Bengal, Darjeeling-734 430, West Bengal, India

Received 16 May 1994, accepted 26 September 1994

Abstract : The analytic solutions, obtained by Riazi and Askari, of the approximate and exact vacuum Brans-Dicke equations for the spherically symmetric static case are shown to correspond to unphysical negative values of the coupling parameter w The present investigation is meant to highlight the pitfalls that one might encounter in the physical interpretation of such solutions.

Keywords : Vacuum Brans-Dicke equations, spherically symmetric static case, analytic solutions

PACS No. : 04.20.Cv

The idea of utilizing the Brans-Dicke (BD) theory in the interpretation of various astrophysical phenomena is quite attractive. With this intention, Riazi and Askari (RA) [1] have recently obtained analytic solutions (eqs. (13), (20) and (22) of [1]) of the approximate BD equations in the spherically symmetric case. We might call these solutions the 'approximate' RA solutions. These solutions have been utilized to interpret a very important astrophysical phenomenon, namely, the observed flat rotation curves in the vast domain of dark galactic haloes. According to the authors, the interpretation requires a rather 'unnatural' large positive value of w ($\sim 10^{12}$). In our opinion, such a requirement by itself does not constitute any inadequacy of the approximate RA solutions as the large mass of astronomical probes indicates only an ascending order of positive values for w (≥ 6, 29, 140, 500, ...). However, it turns out that the RA solutions correspond only to negative values of the BD parameter w. Consequently, some care should be exercised in the application of the RA solutions to those problems of physical interest that require a positive w.

---

*On leave from : Department of Mathematics, Tolaram Government College, Narayanganj, Dhaka, Bangladesh

The BD field equations are :

$$\Box^2\phi = \frac{8\pi}{3+2w} T^{\mu}_{M\mu}, \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{M\mu\nu} - \frac{w}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi_{;\rho}\phi^{\rho}_{,}\right.$$

$$\frac{1}{\phi}\ {}_{;\mu;\nu} - g_{\mu\nu}\Box^2\phi), \tag{2}$$

where $\Box^2 \equiv (\phi;^{\rho})_{,\rho}$ and $T_{M\mu\nu}$ is the matter energy momentum tensor excluding the ϕ-field, w is a dimensionless coupling constant.

Riazi and Askari [1] show that, in the asymptotic region, their approximate solutions, $B(r)$ and $A(r)$, behave as follows (speed of light $c = 1$) :

$$B(r) \rightarrow 1 - \frac{(c_2 - 2)r_0}{r}, \tag{3}$$

$$A(r) \rightarrow \left(1 - \frac{c_2 r_0}{r}\right)^{-1} = 1 + \frac{c_2 r_0}{r} + 0(r^{-2}), \tag{4}$$

where $d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Further, from the above expression for $B(r)$, they identify the total mass M of the configuration (inclusive of the contribution from the ϕ-field) with $\frac{(c_2 - 2)r_0}{2G}$, where r_0 is a constant c_2 is a dimensionless constant of integration. On account of the positivity of energy, we find that two cases are possible : (i) $r_0 > 0$ and $c_2 > 2$, (ii) $r_0 < 0$ and $c_2 < 2$. In their paper, RA consider only case (i) as is reflected in their requirement of fine tuning $c_2 = 2^{+}$. We shall first investigate case (i).

Consider the Robertson expansion in standard coordinates [2].

$$d\tau^2 = \left(1 - 2\alpha GMr^{-1} + 2(\beta - \alpha\gamma)G^2M^2r^{-2} + \ldots\right)dt^2$$

$$- \left(1 + 2\gamma GMr^{-1} + \ldots\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5}$$

This represents spherically symmetric static solutions of BD equations in vacuum [3] provided $\alpha = \beta = 1$, $\gamma = \frac{w+1}{w+2}$, together with

$$\phi = \phi_0\left(1 + (w+2)^{-1} GMr^{-1} + \ldots.\right). \tag{6}$$

In the asymptotic region (*i.e.*, to the first order in r^{-1}), the expressions (3) and (4) must be compatible with the corresponding parts from the expression (5). This requirement should be regarded as a boundary condition to be satisfied by all spherically symmetric solutions of the

BD equations. The physical viability of different solutions can also be judged by the same token. For the approximate RA solutions, we have

$$(c_2-2)r_0=2GM, \tag{7}$$

$$c_2 r_0 = 2GM\left(\frac{w+1}{w+2}\right). \tag{8}$$

Eliminating $2GMr_0^{-1}$ from the above, we get unphysical negative values for w :

$$w = -\left(\frac{c_2+2}{2}\right) < 0 \qquad \text{if } c_2 > 2, \tag{9}$$

which is what we set out to show.

One might suspect that the appearance of unphysical negative values of w is due somehow to the approximate nature of the RA solutions that we have considered. It will soon turn out that this is not so; the negativity of w persists even if the exact BD equations are considered. The spherical solutions of the exact BD equations have been obtained (RA) through a conformal reparametrization procedure [4–6]. We might call these solutions the 'exact' RA solutions and in standard coordinates, they are given by [4]

$$\phi(r) = \phi_0 + \frac{Q_s}{r} + \left(1+\frac{Q_s}{2M}\right)\frac{GMQ_s}{r^2} + \left(\frac{(w-6)Q_s^2}{6M^2} + \frac{11Q_s}{3M} - \frac{8}{3}\right)\frac{G^2M^2Q_s}{2r^3} + \ldots,$$

$$B(r) = 1 - \frac{2GM}{r} + \frac{2G^2MQ_s}{r^2} + \left(\frac{(w-16)Q_s}{6M} + \frac{5}{2}\right)\frac{G^3M^2Q_s}{r^3} + \ldots,$$

$$A(r) = 1 - \frac{2G(Q_s-M)}{r} + \left(\frac{(8-w)Q_s^2}{2M^2} + \frac{9Q_s}{M} + 4\right)\frac{G^2M^2}{r^2} + \ldots, \tag{10}$$

where
$$2M = Q_s(1-\delta) = \frac{r_0}{G}(\delta-1), \tag{11}$$

Q_s and δ are two integration constants. These are related to r_0, c_2, ϕ_0 and c_1 by the relations : $Q_s = -c_1 = -r_0/G = -r_0\phi_0$ and $\delta = c_2 - 1$ so that

$$2M = c_1(c_2-2). \tag{12}$$

From this, we see that $c_2 > 2$ translates to $\delta > 1$ and that eq. (12) is nothing but eq. (7) redefined. Also the so called fine tuning condition is now restated as $\delta = 1^+$. After making this change, the asymptotic expressions for $B(r)$, $A(r)$ and also $\phi(r)$ have to be compared with those from eqs. (5) and (6). In the first order in r^{-1}, these give the equations

$$B(r) \sim 1 - \frac{Q_s(1-\delta)G}{r} = 1 - \frac{2MG}{r}, \tag{13}$$

$$A(r) \sim 1 - \frac{Q_s(1+\delta)G}{r} = 1 + \left(\frac{w+1}{w+2}\right)\frac{2GM}{r}, \tag{14}$$

$$\phi(r) \sim \phi_0 + \frac{Q_s}{r} = \phi_0\left[1 + \frac{GM}{(w+2)r}\right]. \tag{15}$$

From eqs. (13)–(15) that now also include the expression for $\phi(r)$, it follows that

$$w = -\left(\frac{\delta+3}{2}\right) < 0 \qquad \text{since } \delta > 1. \tag{16}$$

Clearly, this result is again the same as eq. (9) above with $c_2 = \delta + 1$. The exact solutions (10) also permit us to go beyond the first order approximation. For instance, consider the second order term in $B(r)$ from eq. (5) and equate it with that from eq. (10) :

$$2(\beta - \alpha\gamma)\frac{G^2M^2}{r^2} = \frac{2G^2MQ_s}{r^2},$$

with $Q_s = \frac{2M}{1-\delta}$. With the specified values of α, β, γ, one immediately finds that eq. (16), or, by another symbol, eq. (9), continues to hold good. We should also recall that the second order term in $B(r)$ plays a crucial role in the Solar system tests of gravity. The most significant one is the wellknown test for the precession of planetary orbits. On the other hand, in the standard BD theory, $w \geq 6$, if all classical tests of General Relativity are to be reasonably accounted for. Thus, eqs. (16) or (9) prevent RA solutions to become applicable in the Solar system scenario, although they may be applicable in other physical situations.

Let us now examine case (ii) : $r_0 < 0$ and $c_2 - 2 < 0$. In this case, it is possible to choose a range of values for c_2 such that a positive w is obtainable from eq. (9). For example, $-\infty < c_2 < -2$ guarantees a positive w. However, we still have to leave out the range $-2 < c_2 < 2$ as this leads to a negative w. Anyhow, everything looks nearly fine as far as w is concerned but the problem crops up elsewhere. The rotational velocity in the range of flat rotation curves becomes imaginary! It is given by $v_{\rm rot}^2 \cong (c_2 - 2)/(2w)^{1/2}$. Thus, once again, we run into a physically meaningless conclusion but of a kind different from that corresponding to case (i).

Riazi and Askari [1] themselves caution against yet another limitation : the vacuum condition may not be tenable in view of the possible presence of luminous matter in the galactic haloes. We agree with their opinion. Probably an interior BD solution ($T_{\mu\nu} \neq 0$) would be closer to the physical situation.

Acknowledgment

One of the authors (AI) would like to thank the Indian Council for Cultural Relations, Azad Bhawan, New Delhi, for a fellowship under an Exchange Programme of the Government of India.

References

[1] N Riazi and H R Askari *Mon. Not. Roy Astron. Soc.* **261** 229 (1993)

[2] S Weinberg *Gravitation and Cosmology* (New York : John Wiley) p 246 (1972)

[3] S Weinberg *Gravitation and Cosmology* (New York . John Wiley) p 183 (1972)

[4] H R Askari and N Riazi *Exact Spherical Solutions of Brans-Dicke Theory in Standard Coordinates*, Publication No. **40** of the Biruni Observatory, Shiraz University, Iran (Private Circulation) (1993)

[5] R H Dicke *Phys. Rev.* **125** 2163 (1962)

[6] T Singh and L N Rai *Gen. Rel Grav.* **15** 875 (1983)